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W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.

AND

PROF. E. T. WHITTAKER, M.A., F.R.S.

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All correspondence concerning the *contents* of the **Gazette** should be addressed to W. J. GREENSTREET, The Woodlands, Burghfield Common, nr. Mortimer, Berks.

Correspondence relative to the **Mathematical Association** or the *distribution* of the **Gazette** should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, 39 Burlington Road, Chiswick, W. 4; Miss M. PUNNETT, B.A., London Day Training College, Southampton Row, W.C. 1.

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UNICURSAL PLANE CURVES.

BY G. B. MATHEWS, F.R.S.

THE theory of unicursal plane curves affords so many interesting exercises that teachers may like to have a short collection of formulae and theorems about these curves, reduced to the shape which appears to be most convenient for applications.

With Salmon's notation we find that when p, the genus of the curve, is zero,

$$\delta=\tfrac{1}{2}(m-1)(m-2)-\kappa. \tag{1}$$

$$\iota=3m-6-2\kappa. \tag{2}$$

$$\begin{aligned}\tau&=2(m-2)(m-3)-\tfrac{1}{2}(4m-11)\kappa+\tfrac{1}{2}\kappa^2\\&=2(m-3)(m-2-\kappa)+\tfrac{1}{2}\kappa(\kappa-1)\end{aligned} \tag{3}$$

$$n=2m-2-\kappa. \tag{4}$$

Suppose now that we have a rational curve of order m, referred to a triangle each side of which cuts the curve in m ordinary points; then, with t as a parameter, we may put

$$x:y:z=\phi(t):\chi(t):\psi(t),$$

where ϕ, χ, ψ are polynomials of order m. We further assume that $t=\infty$ corresponds to an ordinary point on the curve.

Putting $d\phi/dt=\phi'$, $\phi\chi'-\phi'\chi=(\phi\chi')$, etc., the equation

$$(\chi\psi')x+(\psi\phi')y+(\phi\chi')z=0 \tag{5}$$

is what we may call the crude form of the tangent at (t). It involves t to the order $2m-2$, that is, $n+\kappa$, and represents, when determinate, a definite line meeting the curve in at least two consecutive (not merely coincident) points at (t). But if $dv\{(\chi\psi'), (\psi\phi'), (\phi\chi')\}=C(t)$, then the roots of $C(t)=0$ correspond to points such that every line through any one of them meets the curve in two consecutive points. I shall call these the cuspidal points, and $C(t)=0$ the cuspidal equation. It follows from (4) that $C(t)$ is of order κ in t. Dividing (5) by $C(t)$, the result is an equation

$$\Phi(t)x+\chi(t)y+\Psi(t)z=0, \tag{6}$$

which we may call the reduced equation of the tangent, and if we put

$$\xi:\eta:\zeta=\Phi:\chi:\Psi,$$

we have a representation of the curve in tangential coordinates.

The condition for a stationary tangent is

$$\Delta=\begin{vmatrix} \phi & \psi & \chi \\ \phi' & \psi' & \chi' \\ \phi'' & \psi'' & \chi'' \end{vmatrix}=0,$$

the order of which is $3m-6$, which by (2) is the same as $\iota+2\kappa$. Now the stationary tangents are either inflexional tangents or cuspidal tangents, so we infer the identity

$$\Delta=C^2S,$$

where $S=0$, of order ι, determines those points which are to be regarded as inflexions.

It frequently happens that $t=\infty$ corresponds to a singular point, and this may give rise to an apparent reduction of order in the equations $S=0$, $C=0$. We may remove the discrepancy in two ways: either by using a homogeneous parameter (t, u) instead of t and modifying the anaylsis accordingly, or else by putting $t=u^{-1}$ and forming the cuspidal and inflexional equations in u.

An example will illustrate the foregoing remarks. For Bernoulli's lemniscate we may put

$$x:y:z=t^3+t:t^3-t:t^4+1,$$

leading to the crude equation of the tangent

$$(t^2+1)(t^4-4t^2+1)x-(t^2-1)(t^4+4t^2+1)y+4t^3z=0,$$

which is also the reduced equation, so that $\kappa=0$. The equation $\Delta=0$ is found to be

$$t(t^4+1)=0,$$

which is apparently of the fifth order, whereas by (2) it ought to be of the sixth. The reason is that the point $x=y=0$ corresponds to $t=\infty$ as well as to $t=0$, and is therefore a node, and both the tangents there are inflexional. So the proper way of writing $\Delta=0$ is

$$tu(t^4+u^4)=0,$$

and we have six inflexions, two at each of the three flecnodes, which are the singular points of the curve.

Again, take the cardioid, for which we may put

$$x:y:z=1-t^2:2t:(1+t^2)^2;$$

the crude form of the tangent is

$$(1+t^2)[(3t^2-1)x-3t(1+t^2)y+z]=0;$$

this gives $n=3$, and apparently $\kappa=2$. But really $\kappa=3$ by (4), and the missing cuspidal point is $x=y=0$, for which $t=\infty$, and the proper cuspidal equation is

$$u(t^2+u^2)=0.$$

Calculating Δ for this example, we find

$$\begin{aligned}\Delta&=(1+t^2)\{-2(3t^2-1)+(4+12t^2)\}\\&=6(1+t^2)^2,\end{aligned}$$

so $S=1$ and $\iota=0$ in accordance with (2).

Formula (3) gives $\tau=1$, so that the only double tangent is the real one.

G. B. Mathews.

THE LIGHTER SIDE OF MATHEMATICS.

By C. A. Stewart, M.A., Lecturer in Mathematics, University of Sheffield.

(*Paper read before the Yorkshire Branch of the Mathematical Association, at Sheffield, on Saturday, 20th November,* 1920.)

In our profession we often come into contact with those who do not understand Pure Mathematics. Some of these are respectful as if entering a shrine; but others, of the baser sort, are contemptuous. These have found the way of progress dark and difficult. Their guides, perhaps, have not inspired them, and symbolism seems far removed from the needs of ordinary human life. They deny that it can have an ideal, that it can have any claim to beauty. They admit that it has some purpose in its application to the practical affairs of life, but that it can have an end in itself is incomprehensible. It is associated in the minds of some with the removing of interminable brackets, and with the chasing of elusive unknowns. Since it deals usually with the variable and not with the particular, and since it is concerned more with deductions from data than with the truth of these data, it has been described as the subject "in which we never know what we are talking about, nor whether what we are saying is true." * It is usually considered unwise to express lack of appreciation of the works of a great painter or sculptor or poet, but there have been men of intellect who not only have expressed ignorance of scientific method, but have also been inclined to exaggerate that ignorance. The questions that such a criticism of the subject naturally raises are: Is this hostility justified? Can it be wholly attributed to feebleness of insight? For it is possible that although Mathematics need not fail to supply the necessary stimulus to intellectual thoughts and aspirations, yet its exponents may fail in the interpretation and expression of its ideals; and it must be admitted that the way of mathematical learning can be, and often is, a dull and cheerless one, especially for him whose aptitude is weak. The inherent beauty of mathematical reasoning may be marred by methods that are brutal, and its outlook warped by unduly insisting on the necessity of the moment.

My object in this paper is to consider the position of the mathematician in two of his possible moods—during his moments of leisure and during his moments of reflection. Are there pleasant bypaths of mathematical thought and speculation for him to wander in during his hour of freedom? Are there fields of activity within its sphere that can fitly be described as inherently attractive? And is there scope in mathematical teaching for the occasional introduction of what is distinctly recreative or amusing? He has, too, his moments of weakness, his hours of doubt, when he is oppressed by the feebleness of his efforts, the apparent uselessness of all his endeavour and the emptiness of his outlook. Is there sufficient purpose in what mathematicians have achieved to provide him solace at these times? Has the use of Mathematics been so beneficial in the interests of humanity as to strengthen him in the continuation of his task? It is then that his mind becomes receptive of a higher conception of his subject. In contemplating its strength and greatness, its permanence and beauty, he recognises that it has defined his attitude to the world and has guided him in the art of living.

This, then, is the first consideration: Is Mathematics interesting or not? —one not unimportant in the view of the modern trend of educational method. Can the subject be made attractive not merely to those who have the natural aptitude, but also to that greater number who apparently use it as a means towards an end? The justification of the introduction of any element that will have the tendency to drive away monotony, is apparent when we consider how progress is dependent on the awakening and the maintaining of interest.

* Bertrand Russell, *Mathematics and Metaphysics*.

It is naturally impossible for me, in the time at my disposal, to attempt a detailed exposition of the recreative aspect of mathematical work; and so I must be content to give a brief survey of the subject from this point of view, to make a short reference to one or two of the suggestions that will have been made and to make one theme the chief illustration. Mathematics is particularly fortunate in possessing a wide field from which to select work of this description, the following being some of the possibilities: Recreations, Arithmetical, Geometrical and Mechanical; Practical Mathematics; its uses in such spheres as business, sport or war; its applications to other sciences; its history, not only of individuals but of the problems that have been handed down from the past; and lastly, a more recondite subject, hardly suitable for the immature—I mean the study of its fundamental notions, a study that is necessary for obtaining a true conception of its ideals and infinite possibilities.

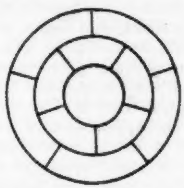
There are three facts in the modern history of Mathematics that show a tendency in the direction of which I am speaking—a tendency towards the lightening of the burden of the mathematical pupil. The first is the use of Analytical methods which became universal in England about ninety years ago; the other two, of comparatively recent date, are the superseding of Euclid and the introduction of Practical Mathematics. The treatment that has replaced Euclid is open to innumerable objections, but it still remains a step in the right direction, and was taken on the continent long before we thought about it. The deadening influence on the youthful mind of abstract methods of exposition is further mitigated by the introduction of Practical Mathematics; and it is therefore not unreasonable to suppose that this subject, which is in its infancy, will in future include sections not hitherto contemplated.

This naturally leads us on to think of the interesting story of the applications of Mathematics to other sciences and to the ordinary business of life. Much of it is known at least vaguely to most people, and it is unnecessary here to illustrate what is common knowledge. It is difficult, however, to think of the subject at all, without at least referring to the remarkable work of that illustrious mathematician and physicist, James Clerk Maxwell. But rather than give positive instances of the extensive use of mathematical results, I venture to describe an incident from the late war to show the kind of situation that arises when the application of a mathematical idea is entrusted to the uninitiated. In the schools of instruction established for officers, many of the classes were, as is well known, conducted by sergeants of the regular army. These instructors were efficient in many ways, but it would be an exaggeration to say that their knowledge of science was profound. One of them I know of was explaining to his class how the angle of elevation was measured, and in the course of his argument made the statement that the diameter of a circle went exactly three times into the circumference. One of his audience was dubious of this, and asked him if that was quite accurate. "Was there not a little bit left over when the division was made?" This made the instructor shaky of his position, and he said he would get the sergeant-major to explain. This particular sergeant-major had the reputation of being omniscient; and, in any case, being a sergeant-major, was never at a loss for a reply. He came over to the class and said he would explain how it was that the diameter went into the circumference exactly three times. Taking a penny from his pocket and holding it up, he said, "Look at the little circle formed by this penny. If you were to measure the diameter and then the circumference, you would find that the one was three times the other." "But, of course," he added disdainfully, making a large sweeping movement with his hands, "If you had a *big* circle, anything might happen."

Some people are above Mathematical recreations, but when we remember that in most of us there is something imaginative, something of the spirit

of emulation, we can afford to ignore their views. Some would say that recreations are too trivial to deserve serious thought, but what has been the study of men like Euler, Fermat and Legendre, we cannot affect to despise. Leibnitz in one of his letters remarked * that men were never more ingenious than in the invention of games; here the mind was at ease; after games that depended solely on numbers came those of position; and after those, where only number and position appeared, came the games that involved motion; that one would, in fact, desire to have a course of study, entirely devoted to games treated mathematically.

Geometrical recreations would consist chiefly of fallacies, paradoxes and games of position, and it will be sufficient for me to illustrate them by taking a typical example of the latter known as the Hamiltonian Game.† This was invented by Sir William Hamilton, and consists in finding a path along the edges of a regular dodecahedron which will include every vertex once and once only. The twenty vertices are given the names of towns, and it is required to find a path from a town A through every other town once and returning to A. A simple way of representing the solid on a plane for the purposes of solution is as follows:



A path satisfying the required conditions may be found by trial, but it is interesting to consider the application of the mathematical method. On entering a town, there are two alternatives—either turning to the right or to the left. Denote the operations of turning to the right or left by the symbols r and l respectively. Then we can find certain laws which these operations satisfy: *e.g.* $r^5 = l^5 = 1$; $rl^2r = lrl$. The problem is therefore reduced to finding a relation containing twenty operations, the total effect of which is unity. Such a relation is: $(l^3r^3(lr)^2)^2 = 1$. The problem is varied by imposing certain restrictions on the order in which the towns may be visited.

The subject of Arithmetical recreations forms part of a section of mathematical work which is sometimes regarded as special in character but whose results permeate the whole of Mathematical study. The Theory of Number is full of paradox and mystery, treating of entities apparently simple in character, but really so complex as to provoke everlasting discussion. Number is the root of all mathematical evil; it is the source of much mathematical pleasure. The study of its true significance has traditionally belonged to abstract philosophy, but now the mathematician can give the more adequate interpretation. The processes of counting and tallying are amongst the first activities of human intelligence; and the problems that arise in connection therewith form the subject-matter of the most recent mathematical research. Its domain is vast. A true conception of its nature is fundamental to Analysis, and the modern theory corresponds in a remarkable way to the traditional and empirical notions of spatial magnitude.

Positive numbers present themselves in a dense array stretching from the infinite to the infinitesimal. Imagine them arrayed in order of magnitude as we do when we represent them by points on a line. They are so dense that an infinity of them exists between any two of them, however near.

* Quoted by W. W. Rouse Ball, *Mathematical Recreations*, p. i. 1919. † *Ib.* p. 189.

But of this vast aggregate certain numbers stand out with great prominence, possessing properties of a special and often mysterious nature. They are human in some of their attributes, for some are perfect and some are amicable; some lucky and some unlucky. One investigator who spent the greater part of his life in studying them became so familiar with them that he was said to regard each positive integer as a personal friend.* Although this class of positive integers forms the subject-matter of a theory by itself, the interesting fact in connection with them is that the solution of many of its classical problems is intimately associated with the theory of functions of continuous magnitude. One feature that distinguishes it from most of the other branches of Mathematics is the comparative simplicity with which its problems can be stated; not merely the easier problems, but even those that have hitherto defied solution. For example: What is the minimum number of squares which when added together will equal any given number? What is the number of primes less than n? Goldbach's Theorem that every even number is the sum of two primes. Waring's general problem and the theory of Mersenne's numbers; and the famous last theorem of Fermat that $x^n+y^n=z^n$ has no solution in positive integers if n is greater than 2. This simplicity in subject-matter makes it a fruitful field for mathematical recreation. The problems are numerous, and many of them may be found in the well-known work of Mr. W. W. Rouse Ball. The questions that arise may be trivial or important; they may make problems of a day or problems for all time. There is a game which may sometimes be seen played for high stakes in the smoke-room of an Atlantic liner, and which may serve as an illustration of the former type. It has the appearance of being solely dependent on the chance arrangements of the numbers concerned, but in reality an expert can nearly always beat an unskilful opponent. The contents of a box of matches are divided at random into three heaps. The rule of the game is that the one whose turn it is to play may take one or all from any heap, but he must not touch more than one heap. The loser is the one who is forced to take the last match. If, for example, after several moves, a player leaves his opponent 1 : 1 : 1, he must win, for his opponent is forced to take the last match. Again, if he leaves 1 : 2 : 3, it is not difficult to see that he can also force a win. The method of the expert is to remember certain key-numbers which will force a win from the initial outlay. The solution of the problem of finding these key-numbers has been given in an American journal,† and is very ingenious. It consists in expressing the numbers in each heap in the binary scale. If the sums of the corresponding digits are all even, the outlay forms a key-number. If some are odd, it can be so altered according to the rules to form the required combination. For example, if the initial outlay were 8 : 7 : 9, these numbers in the binary scale are 1000 : 111 : 1001. The sums of the corresponding digits are 2, 1, 1, 2. It is therefore not a key-number, but can be made one by removing the two 1's, *i.e.* by changing the middle heap to 1.

The above example illustrates the recreative aspect of number, but gives no indication of the vast number of ingenious problems and games of skill that have been devised. The more serious types of problems may not always be justly termed recreative, but they certainly can be described as interesting. They are characterised by their age and historical connections, by the simplicity of their statements and by their intractability. This is sufficient to command the attention and interest of even the immature, and it would not be ridiculous to imagine a school-boy making a praiseworthy attempt at the solution of Fermat's last theorem. There is probably no other branch of scientific work where such a remark would apply. The fact that a great deal of recent mathematical research has been concentrated in this direction

* Prof. G. H. Hardy, *Some Famous Problems of the Theory of Numbers*. 1920. p. 21.

† W. W. Rouse Ball, *Math. Recr.* p. 21.

is an item of additional interest. Progress is being made in two main classes of problems—in the theory of partitions and in the theory of the distribution of primes. In both of these is seen the same fundamental characteristic—the intense difficulty of the analysis combined with the remarkable simplicity of the results aimed at. It is comparatively simple to enunciate the problem: In how many ways is it possible to express a number n as the sum of other numbers less than n? but its solution requires the discussion of an elliptic modular function. It is suggested, then, not that the methods here are simple, but that the results achieved and contemplated are in many cases within the comprehension of those of moderate attainments.

The positive integers, however, form only a very small class of the numbers that exist. The theory of rational and irrational numbers brings problems to our notice of a totally different character. It is here that the mathematician has entered the realm of the philosopher, and has seized territory from which it will be difficult to dislodge him. The problems of the infinite and the infinitesimal have been inadequately represented since Zeno first propounded them in his famous paradoxes. The great Eleatic, unconscious of the false premises latent in his mind, obtained results inconsistent with experience. A strict investigation into the nature of the mathematical infinite, effectively completed by the mathematician, is necessary for the solution of the paradoxes. The fallacy is now seen to consist in applying the properties of finite aggregates to the infinite. That the whole is greater than its part, or that a number can be obtained by the process of counting seem self-evident statements, but they produce disastrous results when applied to the infinite. The accurate theory of the irrational and of the infinite, first propounded by Dedekind and Cantor about fifty years ago, gives rise to many interesting speculations. What, for example, is the correspondence between a set of points and the continuum of real number? Are all infinite numbers equal? Is there a greatest number? It seems natural to suppose that there must be, namely, the number that includes everything, but Cantor proves that it does not exist.

From such speculations the mind naturally passes to the consideration of the general philosophy of Mathematical thought: its purpose and ideals. Its value in a technical education will be revealed by its usefulness, but its ideals alone measure its value in a liberal education. The Professor of Mathematics at Oxford said quite recently that the life of a mathematician was possibly one which no perfectly reasonable man would elect to live, and that "it is something to be able to say that at any rate we do no harm." * There is a chance of such a remark being taken more seriously than was intended, for there is a tendency for the mathematician to concede all that is artistic to the classical and literary scholar. He is apt to think that its manifold uses in the practical life are sufficient to compensate what many regard as its deficiency in culture. But there must be value in a study that tends to regulate the mind in its outlook on life—one freed from those natural prejudices and desires that prevent a clear, uncoloured view of human conduct. The scientific method, as Bertrand Russell has persuasively shown,† is free from the disturbing influence of ethical considerations, and might be oftener used in other branches of thought. There have been instances even in the history of science itself where progress has been hindered by the introduction of ethical and religious beliefs. To believe that the earth went round the sun was to show the greatest irreverence, this being inconsistent with the theory, so comforting to human egoism, that the earth was the centre of the universe. The classicist and literary scholar, we are often told, lives in the past, and when contemplating the things of the present, will exclaim in the words of the Ecclesiast, "There is nothing new under the sun." Talk

* *Some Famous Problems of the Theory of Numbers.* 1920. p. 4.

† *Scientific Method in Philosophy.*

to him of some event in modern history, and he will give you a quotation from the austere historian of Greece, describing an exact parallel in the Peloponnesian War. However that may be, the mathematician is not so encompassed by the presence of a completed system of knowledge, and consequently his endeavours are not obvious repetitions of what has already been done. It may be that the Chinese once knew of Taylor's theorem, or that the chimpanzee was once acquainted with the rule of three, but we certainly do not know these statements to be true; and we think them highly improbable. There is always, therefore, the hope of discovery of something new, and comfort in thinking of possible achievement.

In the modern striving after practical results, there is an aspect of Mathematics that is apt to be ignored. The high value with which we estimate a technical education should not detract from the aesthetic claim of Pure Mathematics based on the beauty of its pure reasoning. It is not a beauty of colour and shade; it is one of order and austerity. There is nothing incongruous in regarding as a work of art a theorem wherein is displayed a simple exposition of the necessary and sufficient.

This suggests another characteristic often absent elsewhere—the characteristic of permanence; and it can be no mean occupation to be an artificer in an edifice, the foundation-stones of which were laid before Pythagoras lived. Here there is work for the mediocre as well as the talented; as Russell puts it: "The edifice of science needs its masons, bricklayers, and common labourers as well as its foremen, master-builders, and architects. In art nothing worth doing can be done without genius; in science even a moderate capacity can contribute to a supreme achievement."*

GLEANINGS FAR AND NEAR.

62. It is, no doubt, a very laudable effort, in modern teaching, to render as much as possible of what the young are required to learn, easy and interesting to them. But when this principle is pushed to the length of not requiring them to learn anything *but* what has been made easy and interesting, one of the chief objects of education is sacrificed. I rejoice in the decline of the old brutal and tyrannical system of teaching, which, however, did succeed in enforcing habits of application; but the new, as it seems to me, is training up a race of men who will be incapable of doing anything which is disagreeable to them. . . .—Mill's *Autobiography*, p. 53 (1873).

63. "Listening to Edward Gordon arguing law is like listening to a piece of what is meant to be mathematics. The demonstration may often fail, the demonstration tone never.—Cockburn's *Circuit Journeys*, p. 386. [E. S. Baron Gordon, Lord of Appeal (1814-1879.)]

64. It is certain that Madame de Genlis made the present Duke of Orleans such an excellent mathematician, that, when he was during his emigration in distress for bread, he taught mathematics as a professor in one of the German universities [? Swiss].—Maria Edgeworth to Mrs. Mary Sneyd, Edinburgh, March 19, 1803, on her visit to Madame de Genlis.

65. And when at Christchurch 'twas thy sport
To rack my brains with sloe-juice port
And lectures out of number;
When Freshman Folly quaffs and sings
While Graduate Dulness clogs its wings
With Mathematic lumber.

—*Poetical Vagaries.* George Colman the Younger: "A Reckoning with Time," v. 5, Longmans, 1814.

* *Science and Culture.*

THE TRACING OF CONICS.

By Prof. E. H. Neville, M.A.

It is hard to understand why the problem of tracing a conic from its equation is usually subordinated to if not identified with the problem, intrinsically more difficult, of calculating axes, or why students are not encouraged to bring their knowledge of geometrical properties of the curve to the drawing-board. The essence of tracing a curve is to render the sketching of it as accurate as may be desired, and I venture to assert that *for this purpose* the discovery of axes and vertices is not worth a tithe of the labour which it demands. From the advice in some of our text-books one would imagine that the ellipse could be drawn accurately from its four vertices alone, the hyperbola from its vertices and its asymptotes, and the parabola from its axis and vertex and one or two other points*. The truth is, that in any but expert hands these details are wofully insufficient; what is wanted is a multitude of points on the curve, and it matters little whether or not the vertices are included. The student finds satisfaction in drawing the curve from its equation and observing the symmetry that appears when the axes are inserted subsequently.

If θ is a given angle, the point $(x+r\cos\theta,\ y+r\sin\theta)$ is on the conic $ax_2+\ldots=0$ if $a(x+r\cos\theta)^2+\ldots=0$, and the midpoint of the chord through (x, y) with direction θ is (x, y) itself if the coefficient of r in the latter equation is zero, that is, if

$$(ax+hy+g)\cos\theta+(hx+by+f)\sin\theta=0.$$

In other words, chords with direction θ have a diameter, and its equation is that just written. Neither this classical argument nor its appreciation depends on a knowledge of the shapes of conics.

Let us suppose the diameters

$$ax+hy+g=0, \quad hx+by+f=0,$$

which bisect horizontal and vertical chords, to be drawn on graph-paper, and let us call these lines h and v. Unless h is itself almost horizontal, the geometrical operation of passing from a given point P to the point P_h which is such that PP is horizontal and is bisected by h, is both quick and accurate; the lines on the paper keep the direction true, and the distance may be measured by a ruler or stepped off by dividers or by means of marks on the edge of a loose sheet of paper. Similarly P_v, the point such that PP_v is vertical and is bisected by v, is readily constructed from P and v unless v is almost vertical. Thus, in general, if a single point P on the conic has been found, two sequences P_h, P_{hv}, P_{hvh}, ... and P_v, P_{vh}, P_{vhv}, ... of points, all of which are on the conic, can be extended to a considerable length with very little trouble and with a high degree of precision. As a rule, the sequences are not cyclic, and there is no theoretical limit to the number of points obtainable from one starting-point, and in practice, if the curve to be traced is in fact an ellipse, the points of one sequence are often seen to mark every part of the contour more and more distinctly.

But the sequences may be cyclic, and if the sequences are acyclic and the curve is not an ellipse, the sequences pass, and usually pass rapidly, beyond the boundary of the paper; divergent sequences may fail to indicate clearly the shape of the accessible parts of the conic, and cyclic sequences may provide fewer points than are desirable. In either event a simple and effective plan

* One of the best elementary books known to me contains the naïve admission that when the vertices and asymptotes of a hyperbola have been marked, "it is advisable by way of corroboration to find some points on the curve". It is not surprising that later the authors are content after a page of calculation on a particular parabola to assert that they can draw the curve "fairly accurately", and it must be confessed that their figure would not substantiate a less modest claim

is to insert the diameters of chords which bisect one of the angles between the axes of coordinates ; these diameters are the lines

$$(a+h)x+(h+b)y+(g+f)=0, \quad (a-h)x+(h-b)y+(g-f)=0,$$

and we will call them f and g. To keep a ruler or the edge of a sheet of paper symmetrically across the axes of reference, under the guidance of the lines on the graph paper, is little more troublesome than to keep it horizontal or vertical, and so from a point P can be found points P_f and P_g. With the four lines h, v, f, g, one point P leads usually not to a finite number of sequences but to an involved ramification with points such as P_{fhvgv}, and points sufficient for the accurate drawing of the curve are obtained without the steps in any one construction being too numerous for the result to be reliable. Examples do occur in which even with four diameters there are points from which the conic can not be traced, but in most of these cases not only is the curve exceptional in its relation to the axes of reference, but the ineffective starting-points are exceptional points of the curve.

For a starting-point the beginner naturally looks along one of the axes of coordinates or along a line parallel to one of them, but this habit, to the extent to which it is bad, is soon outgrown. In favour of looking for intersections of the curve by the diameters h, v, it is to be urged that these intersections correspond to extreme values of y and x, and indicate the scale on which the figure is to be drawn. Moreover, in the case of an ellipse or an undegenerate parabola intersections with h and v necessarily exist, and in the case of the parabola their calculation leads to an equation that is linear, not quadratic. For a hyperbola whose equation is not actually of the form $xy=k$, one of the four lines h, v, OX, OY does give a real point from which a start is possible.

In the case of a hyperbola, two points which it is easy to find geometrically are the points common to the line $2gx+2fy+c=0$ and the pair of lines $ax^2+2hxy+by^2=0$; the pair of lines, being parallel to the pair of asymptotes, is worth construction apart from this special use. For any value of k, this pair of lines joins the origin O to the intersections of $x=-2bk$ with the circle which has its centre at $\{(a-b)k, 2hk\}$ and passes through O, and also to the intersections of $y=-2ak$ with the circle which has its centre at $\{2hk, (b-a)k\}$ and passes through O ; the choice of k together with the choice between the two modes of construction assists to a combination of convenience with accuracy.

To mark one of the points $\{(a-b)k, 2hk\}$, $\{2hk, (b-a)k\}$ serves another purpose, for to say that the angles from OX to an axis of the curve satisfy the equation $\tan 2\theta = 2h/(a-b)$ is to assert that the directions of the axes bisect the angles between OX and the radius to the first of these points or between OY and the radius to the second. It is a capital delusion to suppose that the directions of the axes are plotted best by finding 2θ from tables or by solving the equation for $\tan 2\theta$ as a quadratic equation in $\tan\theta$, and it is an error of the same kind to solve the pair of equations

$$ax+hy+g=0, \quad hx+by+f=0$$

algebraically in order to mark the centre of the curve. To bisect geometrically the angles between the x-axis and the line $y/x=6/13$ is manifestly quicker and is more accurate than to find that the bisectors are $y/x=(-13\pm\sqrt{205})/6$ and to plot these lines from their equations, and the point of intersection of the lines

$$12x-y-8=0, \quad x+24y-29=0,$$

can not be identified as well from its coordinates, 13/17, 20/17, as from the lines themselves.

It is partly because the coordinates of the centre are not assumed to be calculated that it is not for the actual asymptotes but for lines parallel to them through the origin that a geometrical construction is suggested, but the asymptotes should of course be drawn. The property of equal intercepts

can then be utilised for the rapid insertion of additional points on the curve, and an opportunity is given to emphasise that constructions which in theory are exact may in practice be unreliable.

The methods advocated have been tried successfully with students at different stages of development. Before discussing either the reduction of the general equation or any special forms of conic, I have left a class to discover * on the basis of the diametral property alone various forms which numerical examples do yield. At the other extreme, mature students find the careful plotting of six or eight members of a pencil $S_1+\lambda S_2$ to be the work of only one afternoon in the drawing-school, provided that at least one common point is real and accessible. For different values of λ, the diameters $h_1+\lambda h_2$ have a common point H, the diameters $v_1+\lambda v_2$ a common point V, and so on, and these common points must be marked; then for a particular value of λ, one point H_λ other than H on $h_1+\lambda h_2$ and one point V_λ other than V on $v_1+\lambda v_2$ having been calculated, the diameters themselves join H_λ to H and V_λ to V, the intersection of these diameters is the centre C_λ, and $f_1+\lambda f_2$ and $g_1+\lambda g_2$ are the lines joining this centre to F and G: the conic of centres is traced incidentally. In an alternative process the conic of centres is first constructed with great care, and since in favourable cases thirteen † points of this conic are known in advance, this is not difficult; any point C_λ of this conic being taken as centre, the diameters $C_\lambda H$, $C_\lambda V$, $C_\lambda F$, $C_\lambda G$ can be drawn, and the conic found geometrically from one of the common points without reference to the value of λ which determines it algebraically. For a pencil with no real accessible common points, each conic requires its own starting-point, and the work is longer, but not prohibitive.

NOTES ON EXAMPLES.

(1) $9x^2-24xy+16y^2+32x-76y+16=0$.

This is the parabola mentioned in the first footnote; the diameter f is useless, but h, v, g are efficient, and starts may be made either from points on the axes of reference or from the point where $3x-4y=k$ cuts $100x=4k^2+76k+64$ for various values of k.

(2) $(x+2y-2)^2+4(2x-y+1)^2=45$.

This seems ready-made, but although the axes and vertices can be plotted at once, there is no simpler way to mark additional points than by inserting the diameters h, v and developing sequences from the vertices; it is better to use the four vertices independently than to reflect in the axes of the curve.

(3) $4x^2+9y^2=36$.

As a rule a central conic referred to its own axes can be plotted from the vertices alone by means of the diameters f and g, but in this example A_{fgfg} is so near to B that without other starting points only 3 intermediate points in each quadrant are obtained; however, from $(\pm 9/5, \pm 8/5)$ it is easy to find 28 other points well distributed on the contour.

(4) $14x^2-4xy+11y^2-44x-58y+71=0$.

Here h cuts the curve where $y=3\pm\frac{1}{5}\sqrt{142}$ and v where $x=2\pm\frac{1}{5}\sqrt{110}$, and the first six points in each of the sequences formed from these points by means of h and v combine to mark the curve well; almost the same sequences would be found from the points for which $x=0$ and those for which $x=1$.

(5) $6x^2-xy-12y^2-8x+29y-16=0$.

A hyperbola in which the sequences from h and v alone give as many points as can be desired; since the quadratic terms factorise, it is uncommonly easy to find the points in which $8x-29y+16=0$ cuts the curve.

(6) $6x^2-60xy-19y^2+48x+98y-60=0$.

A typical hyperbola, in which h and v alone are almost useless and must be supplemented either by f and g or by the asymptotes.

(7) $(y-x)^2=4(y+x)$.

Even with the four lines h, v, f, g, the curve cannot be traced from any one starting-point, but the very simplicity to which this result is due renders an elaborate process unnecessary. We may trace the curve either by the intersection of $y-x=k$ with $y+x=\frac{1}{4}k^2$ for different values of k, or, to illustrate a different method, always available for a parabola, by observing that the level at which the horizontal distance to a point of the curve from the line $y-x=0$ is k, is the level at which the horizontal distance from the line $y+x=0$ to the line $y-x=0$ is $\frac{1}{4}k^2-k$.

E. H. NEVILLE.

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* Unfortunately the evidence of value in the experiment is always diminished by the students' possession of books.

† The usual nine points and the points H, V, F, G.

MATHEMATICAL NOTES.

569. [L[1]. 16. a.] *Construction for an Ellipse, given a Tangent and the Length* $a+b$.

If P be a point on the ellipse, eccentric angle ϕ, and if ϕO be drawn from the origin at an angle ϕ to the axis of x, meeting the normal at P in Z, it is easily shown that the locus of Z is the circle whose equation is $x^2+y^2=(a+b)^2$.

Now, if from Z, a point on this circle, a perpendicular be drawn to the tangent, meeting it in P, then P is a point on the ellipse. Parallels through P to the axes cut OZ in X and Y, so that $OX=b$, $OY=a$.

Hence the ellipse may be constructed by the usual methods.

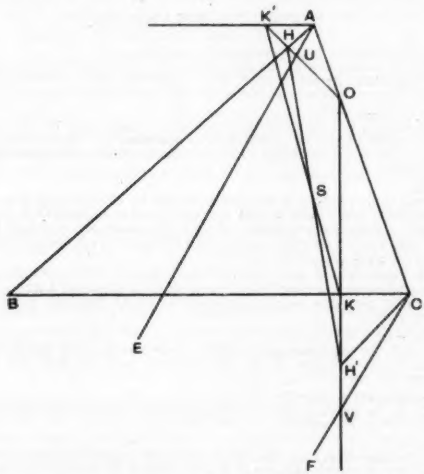
The problem also furnishes us with an easy method of drawing the evolute of an ellipse when the ellipse is given. The ordinary method of drawing an ellipse by bisecting the angle between the focal distances is somewhat inaccurate, since small angles have to be bisected. C. J. SMITH.

570. [K[1]. 1. c.] *The Pillory*, ii. p. 160.

Let O be the fixed point on AC, $OH \perp AB$, $OK \perp BC$.

The pencils $A(U)$, $C(V)$, having corresponding rays parallel, are projective, and hence the rows (U), (V) are projective. They are also perspective, because if AE, CF each coincide with CA, the points U, V coincide at O.

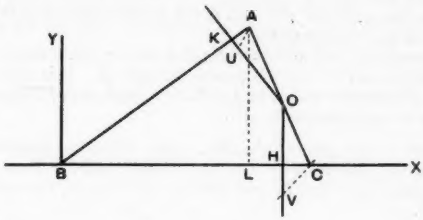
To find the centre of perspectivity, draw AK' parallel to BC meeting OH in K', and CH' parallel to AB meeting OK in H'; then (H, H'), (K, K') are corresponding pairs, and (HH', KK') is the required centre S.



Since KK' goes through S, and $AK' \parallel BC$, the line KK' cannot go through B unless K' is the point at infinity on OH. Similarly H' must be the point at infinity on OK. Hence OH, OK are respectively parallel to BC, BA, and the angle at B is a right angle.

If ABC is not a right angle, the line BS will meet OK, OH in points U_1, V_1, such that AU_1, CV_1 are parallel; that is, in general there is one pair of parallels making B, U, V collinear.

The problem is a good example of cases in which pure geometry is a more effective instrument than analytical.



We may take
$$C=(a,\ 0),$$
$$O=(a-h,\ k),$$
$$A=(a-mh,\ mk).$$

Equation of AV is
$$y-mk=\lambda(x-a+mh).$$

Equation of OK is
$$(a-mh)(x-a+h)+mk(y-k)=0,$$

and that of BV is
$$\frac{y-\lambda x}{mk-\lambda a+\lambda mh}=\frac{mky+(a-mh)x}{(a-mh)(a-h)+mk^2}, \quad \text{(i)}$$

i.e. $\{(a-mh)(a-h)+\lambda(a-mh)mk+m(1-m)k^2\}y$
$$=\{mk(a-mh)+(m-1)(a-mh)h\lambda+mk^2\lambda\}x. \quad \text{(ii)}$$

Equation of OH is $\quad x-(a-h)=0.$

Equation of CV is $\quad y-\lambda(x-a)=U;$

$\therefore$ equation of BV is
$$\frac{x}{a-h}=\frac{\lambda x-y}{\lambda a}$$
or
$$h\lambda x+(a-h)y=0. \quad \text{(iii)}$$

If (ii) and (iii) are the same line,
$$h\lambda\{(a-mh)(a-h)+m(1-m)k^2+(a-mh)mk\lambda\}$$
$$+(a-h)\{mk(a-mh)+(m-1)(a-mh)h\lambda+mk^2\lambda\}=0,$$

which reduces to
$$(a-mh)[mk(a-h)+m\{k^2+(h+k)(a-h)\}\lambda+mhk\lambda^2]=0,$$

i.e. $\quad m(a-mh)[k(a-h)+\{k^2+(h+k)(a-h)\}\lambda+hk\lambda^2]=0,$

that is, $\quad m(a-mh)(k+h\lambda)(a-h+k\lambda)=0.$

This is satisfied independently of λ if
$$a-mh=0,$$
i.e. if B is a right angle.

If we put $k+h\lambda=0$, this means that AU, CV lies along AC, so that U, V coincide at O: this is not a proper solution.

If we put
$$a-h+k\lambda=0,$$
this means that AU, BV are both perpendicular to BO; unless $BO\perp AC$, this gives a proper solution.

G. B. MATHEWS.

The question would be accurate if it had read "any two points of AC," or else "a general point of AC." UBV is always straight for one position of the point, and then UV is $\perp$ to AC.

For, let X be the point on AC, and O the orthocentre of ABC; let AO, CO meet CV, AU in Q, P respectively.

Then, by parallels, $PU : UA = CX : XA = CV : VQ$. Hence COP, AOQ, UV are concurrent, and UV always passes through O. If it also passes through B, then either B coincides with O and ABC is a right angle, or else UV coincides with BO and is at right angles to AC. H. P. H.

Let P denote "any point" in AC. For different positions of P, UV passes through the orthocentre O of the triangle ABC. Hence, when O and B do not coincide, there is still *one* position of P in AC for which UV passes also through B. W. J. DOBBS.

AB, CD are given parallel lines.

From any point P in AC, draw PM, PN *in given fixed directions* to cut AB, CD in M, N respectively.

MN must pass through a fixed point.

If MP cut CD in U, and $CLE \parallel UPM$ cut MN in E and PN in L,

$$\frac{CL}{LE} = \frac{UP}{PM} = \frac{CP}{PA}; \quad \therefore AE \parallel PN;$$

$\therefore$ E is fixed; $\because$ CE, AE are parallel to given fixed directions.

(AE and CE are special positions of MN.)

If $AM = x$, $CN = \xi$, the line is known when x, ξ are known.

You might call x, ξ tangential coordinates of M, N, and a linear relation between x and ξ signifies "MN passes through fixed point."

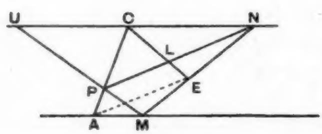


FIG. 1.

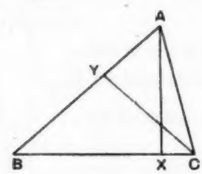


FIG. 2.

In the B.Sc. problem, when it is recognised that UV passes through a fixed point, AX, CY being special positions, UV always passes through the orthocentre. If the orthocentre is B, $ABC = 90°$.

If $x\xi = K^2$, envelope of MN is an ellipse, AB, CD being parallel tangents at ends of diameter AC.

The point of contact of MN and curve cuts MN as $-dx/d\xi$, *i.e.* as x to ξ.

R. F. DAVIS.

571. [I. 2. δ.] *The Bee and the Pentagon.*

In matters hexagonal the bee is our oldest teacher: but her connection with the regular pentagon is unsuspected by many naturalists, and very likely by herself.

The female bee has two parents, like man and other respectable animals; but the male is a parthenogenetic product, and owns a mother only. From this it follows that the number of ancestors of n generations back possessed by the ordinary worker, or her sister the queen, is

$$\frac{2^{n+2}}{\sqrt{5}}(\cos^{n+2} 36° - \cos^{n+2} 108°).$$

W. HOPE-JONES.

572. [M^1. a.] *Area of Cycloid.*

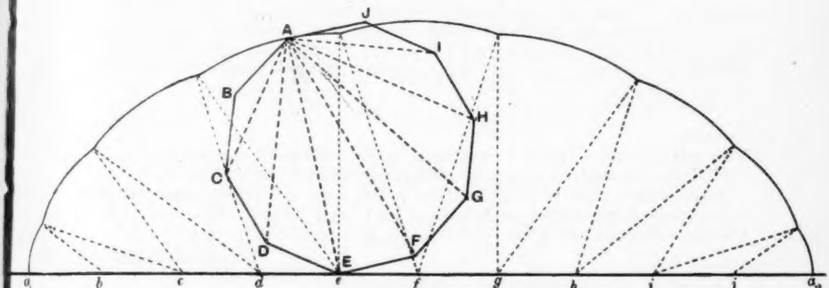
The figure shows the locus of a corner A of a regular polygon of $2n$ sides which rolls along a straight line.

Its area is composed of triangles and sectors of circles.

The triangles are successive positions of the $\triangle$s $ABC, ACD, \ldots AHI, AIJ$.

$$\therefore \text{ their sum} = \text{the area of the polygon.}$$

$$\text{Sum of sectors} = \frac{\pi}{2n}(AB^2 + AC^2 + \ldots + AI^2 + AJ^2).$$



But
$$\left.\begin{aligned} AB^2 + AG^2 &= \text{Diameter}^2, \\ AC^2 + AH^2 &= \text{Diameter}^2, \end{aligned}\right\} \text{etc.};$$

$$\therefore \text{ Sum of sectors} = \frac{\pi}{2n} \cdot n \text{ Diameter}^2$$
$$= 2\pi \text{ Radius}^2.$$

$\therefore$ Area of a loop of the locus = Polygon + twice circumscribing circle.

By increasing n indefinitely,

Area of cycloid = 3 times area of rolling circle.

The radius of curvature is easily deduced from the same figure.

W. Hope-Jones.

573. [D. 2. b.] Cf. Note 549, p. 135. The formula given by Mr. Rickets gives the sum of an H.P. beginning with $1/a$, not $1/(a+d)$ as might be supposed from the third paragraph. It is essentially only the first (or definite integral) term of the modified Euler-Maclaurin formula (*i.e.* with *central* ordinates) which in this case seems to give the complete solution, viz.

$$1/a + 1/(a+d) + \ldots + 1/(a + \overline{n-1}\, d)$$
$$= \int_{-\frac{1}{2}}^{n-\frac{1}{2}} 1/(a+xd) \,.\, dx - \frac{1}{24}\frac{d}{dx}\Big[\quad\Big]_{-\frac{1}{2}}^{n-\frac{1}{2}}$$
$$+ \frac{7}{5760}\frac{d^3}{dx^3}\Big[\quad\Big]_{-\frac{1}{2}}^{n-\frac{1}{2}} \ldots$$
$$= \{\log_{10}(a + \overline{n-\tfrac{1}{2}}\, d) - \log_{10}(a - \tfrac{1}{2}d)\}/(0{\cdot}43429\ldots d)$$
$$- \frac{1}{24}\{1/(a + \overline{n-\tfrac{1}{2}}\, d)^2 - 1/(a - \tfrac{1}{2}d)^2\}$$
$$+ \frac{7}{960}\{1/(a + \overline{n-\tfrac{1}{2}}\, d)^4 - 1(a - \tfrac{1}{2}d)^4\} \ldots.$$

In the example given the second term is $-0{\cdot}0004169$, which virtually accounts for the error shown by Mr. Rickets' formula. [*Cf.* De Morgan, *Diff. and Int. Calc.* p. 311, where he applies the ordinary Euler-Maclaurin formula to the case of an H.P.]

Mr. Rickets' approximate formula was given (for the case $a=1$, from which the general case is immediately derivable) by the late Sir George F. Hardy, K.C.B., in his actuarial classes, *circa* 1888, and published in 1889 in *Graduated Exercises and Examples*, by T. G. Ackland, F.I.A., and G. F. Hardy. The proof is based on the well-known series

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \dots\right\};$$

whence

$$\log_e \frac{1+(n+\frac{1}{2})i}{1+(n-\frac{1}{2})i} = 2\left\{\frac{i}{2+2ni} + \frac{1}{3}\left(\frac{i}{2+2ni}\right)^3 + \dots\right\}$$
$$= \frac{i}{1+ni}$$

very nearly when i (which corresponds to Mr. Rickets' a/d) is small.

Giving n successive values and adding, the result follows at once.

Boole, *Finite Differences*, 3rd edition, p. 103, says:—"Closely connected with the subject of differential coefficients of $\log \Gamma(x)$ is that of ... $\Sigma\{a+(n-1)d\}^{-r}$. On this see papers by Knar (Grunert, xli. and xliii.)."

G. J. Lidstone.

A good second approximation is got by adding

$$\frac{\delta^2}{24z} \quad \text{to} \quad z+\frac{\delta}{2} \quad \text{and} \quad \frac{\delta^2}{24a} \quad \text{to} \quad a-\frac{\delta}{2}$$

in Mr. Rickets' formula. In his special case it reduces the error from

$$+411 \,.\, 10^{-6} \quad \text{to} \quad +18 \,.\, 10^{-6}.$$

In most cases, of course, the first approximation is quite good enough.

W. Hope-Jones.

574. [X. 10]. *An Odd Method for determining the Year of Birth.*

In the second edition of Cajori's *History of Mathematics*, 1919, page 330, there appears an interesting biographical sketch of Augustus De Morgan, the second sentence of which is as follows: "For the determination of the year of his birth (assumed to be in the nineteenth century) he proposed the conundrum, 'I was x years of age in the year x^2.'" It may be of interest to note that after the year 1936 the conditions here given are insufficient to determine the year of birth of De Morgan, since

$$1806+43=(43)^2 \quad \text{and} \quad 1892+44=(44)^2.$$

Perhaps the interest in the given remark, which is said * to have been made by De Morgan, is enhanced by the observation that in every later century there is no more than one year, such that by adding an integer x to it there results a sum which is equal to x^2. For instance, if a man born in the twentieth century will be x years old in the year x^2, he must be born in 1980, since

$$1980+45=(45)^2.$$

For the twenty-first and the twenty-second centuries the corresponding years are 2070 and 2162 respectively, since

$$2070+46=(46)^2 \quad \text{and} \quad 2162+47=(47)^2.$$

* [v. *Budget of Paradoxes*, p. 332; 2nd ed. p. 124, vol. ii. All that De Morgan says is, "I was x years old in A.D. x^2; not 4 in A.D. 16, nor 5 in A.D. 25, but still in one case under that law.]

The thirty-third century is the first century which does not contain a year such that if you add x to it you obtain the year x^2, and the fifteenth century is the next to the nineteenth in which there are two such years; viz. 1406 and 1482, since

$$1406+38=(38)^2 \quad \text{and} \quad 1482+39=(39)^2.$$

These very elementary observations tend to show that if De Morgan made the said remark it does not exhibit his usual thoughtfulness and accuracy.

G. A. MILLER.

575. [K¹. 1. d.]. *To show* $\Delta=\sqrt{s(s-a)s-b)(s-c)}$.

The following trigonometrical proof seems so obvious that it must almost certainly have appeared in some publication, but I have never seen it.

If ABC be the triangle, and I_aZ be the perpendicular from ex-centre I_a on AB produced, then, with usual notation,

$$\tan\frac{B}{2}=\frac{\Delta}{s(s-b)},$$

$$\cot\frac{B}{2}=\cot I_a=\frac{Z_aI_a}{BZ_a}=\frac{r_a}{s-c}=\frac{\Delta}{(s-c)(s-a)};$$

$$\therefore \frac{\Delta}{s(s-b)}\cdot\frac{\Delta}{(s-c)(s-a)}=1 \quad \text{or} \quad \Delta=\sqrt{s(s-a)(s-b)(s-c)}.$$

Charterhouse. N. J. CHIGNELL.

576. [V. s.] *Lagrange's Tribute to Maclaurin.*

In the recent work by Professor Cajori on the *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, the author preludes his remarks upon Colin Maclaurin's *Treatise of Fluxions* with the following statement:

"Maclaurin's book on fluxions has been considered the ablest and most rigorous text of the eighteenth century. It was pronounced by Lagrange, "le chef d'œuvre de géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux."

And there is a footnote "*Mém. de l'Acad. de Berlin*, 1773; quoted in the art. 'Maclaurin' in Sidney Lee's *Dict. of National Biography*."

The statement by Lagrange occurs in his memoir, "Sur l'attraction des Sphéroides Elliptiques (*Mém. de l'Acad. de Berlin*, 1773, p. 121).

Here is the passage: "M. Maclaurin qui a le premier résolu ce problème dans son excellent Pièce sur le flux et reflux de la mer, couronnée par l'Académie des Sciences de Paris en 1740, a suivi une méthode purement géométrique, et fondée uniquement sur quelques propriétés de l'ellipse et des sphéroides elliptiques: et il faut avouer que cette partie de l'ouvrage de M. Maclaurin est un chef-d'œuvre de Géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux."

Clearly this has nothing whatever to do with the *Treatise of Fluxions* of Maclaurin, and it is most unfortunate that such a sublime appreciation of the work of one great mathematician by another should be so cruelly misapplied by two such eminent authorities. There may have been confusion in the mind of the earlier writer for the *D.N.B.* between *flux* and *fluxion*, but such an excuse cannot be tendered for a scholar of Professor Cajori's standing.

The passage from Lagrange is correctly quoted in the *Life of Maclaurin* published in the *Mathematical Gazette*, October 1916, and is to be found also in Chasles' *Aperçu Historique*.

Maclaurin was, of course, "the creator of the theory of the attraction of ellipsoids."

C. TWEEDIE.

577. [L^1. 16. a.] *A chord of a conic passes through a fixed point. To find the locus of its middle point.*

A chord of a conic is always parallel to the polar of its middle point, for the polar meets it in the harmonic conjugate of the middle point, that is, at infinity.

The gradient of the polar of (x, y), the middle point of a chord, is $-\dfrac{X}{Y}$, and the gradient of the chord, if (h, k) is a fixed point on it, is $\dfrac{y-k}{x-h}$.

Hence the required locus is given by equating these, and the result is

$$(x-h)X+(y-k)Y=0$$

or
$$xX+yY+Z=hX+kY+Z,$$

i.e.
$$S=P,$$

where $S=0$ is the conic and P the polar of the given fixed point.

N. M. Gibbins.

578. [X. 4.] *A Diagrammatic Representation of certain Functions of Two Variables.*

I. A relation between three variables x, y, z, of the form $z=f(x, y)$, can in certain cases be represented by a diagram in two dimensions. Take a pair of rectangular axes XOY, ZOZ' (Fig. 1), and let +ve and −ve values of x be set off along OX, OY respectively; +ve and −ve values of y, along OY, OX respectively; and +ve and −ve values of z, along OZ, OZ'. Let (i) and (ii) be any two curves, which we term the "Base Curves," and suppose the ordinates AP of (i) to be given by $AP=P(x)$, and the ordinates BQ of (ii) to be given by $BQ=Q(y)$.

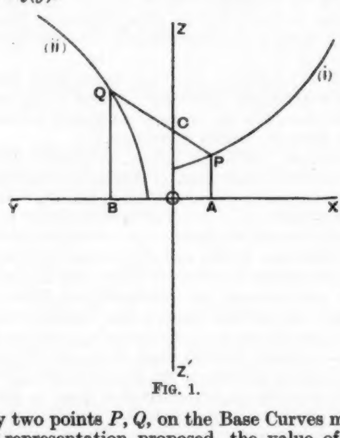


Fig. 1.

If the join of any two points P, Q, on the Base Curves meets OZ in C, OC is, in the method of representation proposed, the value of z corresponding to $x=OA$, $y=OB$. We have

$$OC=\frac{OB \,.\, AP+OA \,.\, BQ}{OA+OB},$$

i.e.
$$z=\frac{yP(x)+xQ(y)}{x+y};$$

$$\therefore\ z\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{x}P(x)+\frac{1}{y}Q(y). \qquad \text{(I.)}$$

We can therefore exhibit the relation $z=f(x, y)$ by means of Fig. 1, provided functions P, Q can be found so that equation (I.) is equivalent to $z=f(x, y)$.

II. (*a*) Consider the function $z=(x^2+y^2)/\{(x+y)xy\}$.

Here $z\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{x^2}+\frac{1}{y^2}$.

Hence, by Equation (I.), $P(x)\equiv\frac{1}{x}$; $Q(y)\equiv\frac{1}{y}$.

(*b*) $z=\frac{1}{x}-\frac{1}{y}$. Here $z\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{x^2}-\frac{1}{y^2}$.

Hence, by Equation (I.), $P(x)\equiv\frac{1}{x}$; $Q(y)\equiv-\frac{1}{y}$.

(*c*) The function $z=xy$. It was the problem of representing this conveniently, in the particular case $S=kNT$ (where $S\equiv$ Shaft Horse-power, $N\equiv$ Revolutions of shaft, $T\equiv$ Reading of Torsion meter, $k=$ Constant), that first suggested the general method.

If $z=xy$, $\qquad z\left(\frac{1}{x}+\frac{1}{y}\right)=x+y$.

Hence, $\qquad P(x)\equiv x^2$; $Q(y)\equiv y^2$.

The Base Curves in this case are therefore the two halves of a parabola (Fig. 2).

If Revolutions of Shaft be set off along OX, Readings of Torsion Meter along OY, and Shaft Horse-power along OZ, a suitably sized parabola will enable the horse-power to be read off for any given values of Revolutions and Torsion.

Or, if pressure and volume of a gas be measured off respectively along OX, OY, and absolute temperature along OZ, a suitable parabola will exhibit the Perfect Gas Law $pv=RT$.

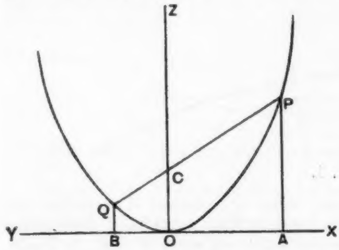


FIG. 2.

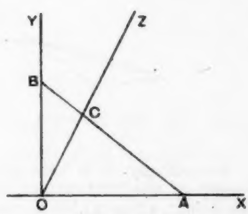


FIG. 3.

III. A relation of the form $\chi(z)=\phi(x)\,.\,\psi(y)$, where ϕ, ψ, χ are any given functions, can be represented by our diagram, provided the axes OX, OY, OZ are respectively graduated proportionally to ϕ, ψ, χ, and the parabola is used as Base Curve. For example, consider the relation $\cos h=\pm\tan l\,.\,\tan\delta$, giving the hour-angle (h) of setting or rising of a celestial body, of declination δ, in latitude l. Graduate OX, OY proportionally to tangents, and OZ proportionally to cosines. Then, with a parabolic Base Curve, we can read off the apparent time of sunset or sunrise for any given declination and latitude.

IV. If, instead of drawing Base Curves corresponding to functions P, Q, we graduate (Fig. 3) OX proportionally to $P(x)$, and OY proportionally to $Q(y)$, so that if the points A, B are marked 'x' and 'y',

$$OA=P(x),\quad OB=Q(y),$$

and take a line OZ of direction cosines l, m, we can exhibit relations of the form

$$1/z = l/P(x) + m/Q(y);$$

for evidently
$$\frac{1}{OC} = \frac{l}{OA} + \frac{m}{OB}.$$

More generally, if OZ be suitably graduated, we can represent

$$1/R(z) = l/P(x) + m/Q(y).$$

E. R. HAMILTON, B.A., B.Sc.

579. [C^1. 1. a; f.] *Note on the differentiation of* $\sin x$, *and on the limit of* $\frac{\sin x}{x}$ *as* x *tends to zero.*

In elementary trigonometry the fact that θ is greater than $\sin\theta$, and that these quantities tend to become equal as θ decreases, is naturally deduced from the appearance of a circular arc, and in elementary calculus the same method is usually adopted, or it is shown that a length can be assigned to the arc.

This is at the best a roundabout way of obtaining the differential of $\sin\theta$, and unless the matter is carefully worded it is made to appear that the value of $\frac{d\sin\theta}{d\theta}$ depends on our arbitrary choice of $2\pi r$ as the length to be assigned to the circumference of a circle.

For regarding the circle as the limit of a straight line path, we can clearly make its length anything we please.

E.g., regarding the length as the limiting distance traversed by a man walking round the circle when his steps become indefinitely short, we get $2\pi r$ as its value, but if we take the limiting distance traversed by a skater who strikes out alternately to left and right at an angle of 60°, we get $\frac{4\pi r}{\sqrt{3}}$ as the length. (Figure 1.)

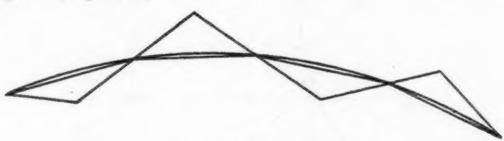


FIG. 1.

Accordingly, it is here proposed that, either (1) instead of assuming the length of the circular arc, it should be assumed that $\sin\theta$ has a differential coefficient, a fact which no one who has used a table of sines is likely to doubt, or that (2) instead of proving the property of the circle, it should be proved that the differential coefficient exists.

Thus, (1) assuming $\frac{d\sin\theta}{d\theta}$ exists, let XOA, XOP be two acute angles containing θ_1 and θ_2 degrees. (Figure 2.)

Give these angles equal increments $\delta\theta$ forming the angles XOB, XOQ. Take OA, OB, OP, OQ of unit length. Join AB and PQ; then these lines are equal. Draw BC and $QR \perp$ to OX to meet AC and $PR \parallel$ to OX.

Then $QR \equiv \delta\sin\theta_2 = PQ\cos PQR = PQ\cos(\theta_2 + \tfrac{1}{2}\delta\theta)$.

Hence we have
$$\frac{\dfrac{\delta\sin\theta_2}{\delta\theta}}{\dfrac{\delta\sin\theta_1}{\delta\theta}} = \frac{\delta\sin\theta_2}{\delta\sin\theta_1} = \frac{\cos(\theta_2 + \frac{1}{2}\delta\theta)}{\cos(\theta_1 + \frac{1}{2}\delta\theta)}.$$

But with obvious notation the limits of these fractions are

$$\frac{\frac{d \sin \theta_2}{d\theta}}{\frac{d \sin \theta_1}{d\theta}} \quad \text{and} \quad \frac{\cos \theta_2}{\cos \theta_1}.$$

So, unless $\frac{d \sin \theta}{d\theta}$ is everywhere zero or infinite, which is obviously not the case, we can put $\frac{d \sin \theta}{d\theta} = \frac{\cos \theta}{k}$, where k is constant.

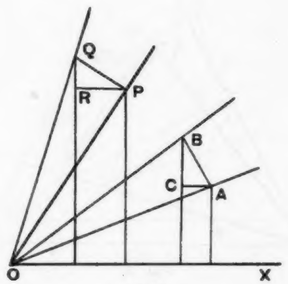


FIG. 2.

Now, taking as unit of angle k degrees and denoting an angle in this measurement by x, we have

$$\frac{d \sin x}{dx} = \cos x;$$

and putting $\frac{90}{k} - x$ for x,

$$\frac{d \cos x}{dx} = -\sin x.$$

Further—for acute angles—$\cos x$ and $\sin x$ are positive, also $\sin x$ and $1 - \cos x$ are zero when x is zero, so $\left(\text{as } \frac{d \sin x}{dx} \text{ is } \cos x\right)$, $\sin x$ increases from zero with x, but more slowly, since $\cos x$ is less than unity. Hence $x > \sin x$.

Again, $\frac{d\frac{x^2}{2}}{dx} = x, \quad \frac{d\,(1 - \cos x)}{dx} = \sin x,$

so $\frac{x^2}{2}$ increases faster than $1 - \cos x$,

or $\frac{x^2}{2} > 1 - \cos x,$ *i.e.* $\cos x > 1 - \frac{x^2}{2}.$

Continuing, $\frac{d\frac{x^3}{6}}{dx} = \frac{x^2}{2}, \quad \frac{d\,(x - \sin x)}{dx} = 1 - \cos x,$

so $\frac{x^3}{6} > x - \sin x$ or $\sin x > x - \frac{x^3}{6},$

and so on, giving as many such inequalities as may be desired, from which, by finding limits for $\sin x$ when x is, say, ·01, we can find k as accurately as our tables permit.

(2) To show that $\frac{\delta\sin\theta}{\delta\theta}$ approaches a limit as $\delta\theta$ decreases to zero.

In Figure 3, XOP is θ degrees, POQ is $\delta\theta$, OP and OQ are of unit length. Imagine the angle POQ divided into n equal parts by lines

$$OQ_1, OQ_2, \dots OQ_{n-1}$$

of unit length. Join PQ, PQ_1, and draw $Q_1N \perp$ to QP.

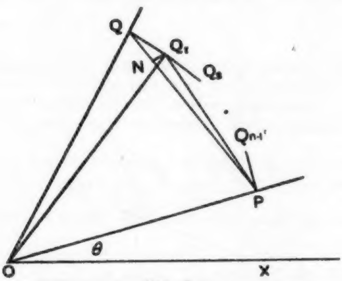


FIG. 3.

Then QN is less than $\frac{1}{n}QP$, for it is the projection on QP of QQ_1, and QQ_1, PQ_{n-1} are the most inclined to QP of the n equal lines QQ_1, $QQ_2 \dots Q_{n-1}P$, whose projections make up QP.

Hence Q_1P, which is greater than NP, is still greater than $\frac{n-1}{n}QP$, or calling angle POQ_1, $\delta_1\theta$, we have $\frac{Q_1P}{QP}$ greater than $\frac{\delta_1\theta}{\delta\theta}$; still more then is

$$\frac{Q_1P\cos(\theta+\frac{1}{2}\delta_1\theta)}{QP\cos(\theta+\frac{1}{2}\delta\theta)} > \frac{\delta_1\theta}{\delta\theta}, \quad \textit{i.e.} \text{ from Figure 2}$$

$$\frac{\delta_1\sin\theta}{\delta\sin\theta} > \frac{\delta_1\theta}{\delta\theta} \quad \text{or} \quad \frac{\delta_1\sin\theta}{\delta_1\theta} > \frac{\delta\sin\theta}{\delta\theta}.$$

Accordingly, as we can take n as large as we please, or, what is the same thing, angle QOQ_1 as small as we please, we see that $\frac{\delta\sin\theta}{\delta\theta}$ continually increases as $\delta\theta$ diminishes.

[In particular, if we put $\theta=0$, so that $\delta\sin\theta$ becomes $\sin\delta\theta$, we have, omitting the δ's, $\frac{\sin\theta}{\theta}$ continually decreases as θ increases from 0 to a right angle.]

This is sufficient for our purpose, for in (1) we saw that if $\frac{\delta\sin\theta}{\delta\theta}$ tended to infinity anywhere it must do so everywhere, but we may show that if $\delta\theta$ is less than 1°, $\frac{\delta\sin\theta}{8\theta}$ is less than $\frac{1}{44}\cos\theta$.

For, in Figure 4 draw $PT \perp$ to OP, meeting OQ in T. Produce PT and mark off $(n-1)$ parts TT_1, $T_1T_2, \dots$ $T_{n-2}T_{n-1}$ equal to PT, where n is the

greatest integer in $\frac{45}{\delta\theta}$, so that, as $\delta\theta$ (POQ) is less than 1, we have $n\,\delta\theta$ greater than 44. Also, as each line TT_1, TT_2 ... subtends a smaller angle at O than does PT, the angle POT_{n-1} is less than 45°, or PT_{n-1}, *i.e.* nPT, is less than 1.

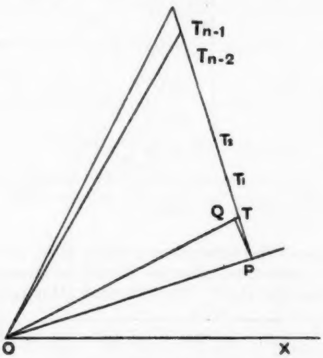


FIG. 4.

Further, considering the angles at Q and T, we find PQ less than PT. Hence we have $PQ \cos\theta$, and still more $PQ\cos(\theta+\frac{1}{2}\delta\theta)$, which, from Figure 2, is $\delta \sin\theta$, less than $PT\cos\theta$, or $n\delta\sin\theta$ less than $\cos\theta$, whence $\frac{\delta\sin\theta}{\delta\theta}$ is less than $\frac{1}{44}\cos\theta$ if $\delta\theta$ is less than a degree.

(3) Finally, if we wish in this way to find the limiting length of the perimeter of a polygon inscribed in a circle of radius r when its sides become indefinitely small, we find that if θ_1 and θ_2 are the greatest and least angles subtended by sides at the circumference, and $\sin\theta_1=\frac{\theta_1}{k_1}$, $\sin\theta_2=\frac{\theta_2}{k_2}$, the perimeter lies between $\frac{360}{k_1}r$ and $\frac{360}{k_2}r$, both of which have for limit $\frac{360}{k}r$, where k is the same number as in (1). C. H. HARDINGHAM.

580. [R. 7. f.] *An Easy Treatment of the Simple Pendulum.*

It is hoped that the following treatment will be found a little more convincing than that usually given in a first course. The aim is to show how the actual motion differs from but approximates to Simple Harmonic Motion; and the treatment does not involve any knowledge of the Calculus.

1. *The Actual Motion.*

With the usual notation it should first be shown that when the pendulum is at A,

$$\text{K.E.}=0,\quad \text{P.E.}=mgl(1-\cos\alpha),$$

and when the pendulum is at P,

$$\text{K.E.}=\tfrac{1}{2}mv^2,$$
$$\text{P.E.}=mgl(1-\cos\theta);$$
$$\therefore\ \tfrac{1}{2}mv^2=mgl(\cos\theta-\cos\alpha);$$
$$\therefore\ v^2=2gl(\cos\theta-\cos\alpha). \qquad (1)$$

Also the accel. along the tangent at P is

$$g \sin \theta. \quad\quad\quad (2)$$

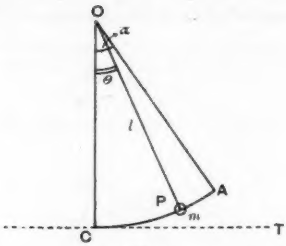


FIG. 1.

Hence the velocity and the accel. *at any point of the arc AC* can be found.

To illustrate this a numerical example should be worked: *e.g.* let $l=2$ ft., $\alpha=40°$, and take $g=32$ f.p.s. p.s.* The following table can easily be compiled:

θ	40°	30°	20°	10°	0°
Vel.	0	3·58	4·71	5·29	5·47
Accel.	20·57	16·00	10·94	5·56	0

It should then be made clear that no easy relation involving TIME, however, can be found; and the pupil should be told that the velocity and the accel. *after any given time*, and even the total time of a complete swing, can only be accurately found by the use of the Calculus.

He may be given the formula

$$T=2\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\frac{9}{64}\sin^4\frac{\alpha}{2}+\dots\right]$$

(see Loney's *Dynamics of a Particle*, p. 106)

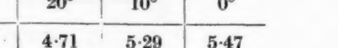
to work out as an example the periodic time in the previous numerical illustration. This time will be found to be 1·62 secs. approx.

2. *A First Approximation to the Actual Motion.*

If the point P had been moving along a straight line of length $l\alpha$ instead of a curve, and *if* the accel. at P had been $g\theta$, the motion would have been S.H.M.

For accel. $=g \,.\, \theta$

$$=g \,.\, \frac{CP}{l}.$$



Hence consider the S.H.M. in which accel. at P is $\frac{g}{l}\,.\,CP$ and the amplitude $l\alpha$. It can easily be shown that vel. would be given by

$$v^2=\frac{g}{l}(l^2\alpha^2-CP^2).$$

* A more accurate value may of course be used, but the important thing is the *comparison* of the numerical results.

Hence, when the particle has still to go $\frac{\theta}{\alpha}$ of the way from A to C,

$$v^2 = \frac{g}{l}\left(l^2\alpha^2 - \frac{\theta^2}{\alpha^2}.l^2\alpha^2\right) = gl(\alpha^2 - \theta^2) \quad \text{.........................(3)}$$

and
$$\text{accel.} = \frac{g}{l}\cdot\frac{\theta}{\alpha}.l\alpha = g\theta. \quad \text{...(4)}$$

These should be compared with results (1) and (2).

For example,

$$2(\cos\theta - \cos\alpha) = 2\left[1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \ldots - 1 + \frac{\alpha^2}{\lfloor 2} - \frac{\alpha^4}{\lfloor 4} + \ldots\right]$$

$$= (\alpha^2 - \theta^2) \text{ as a first approx.,}$$

but the L.H.S. is really slightly less than the R.H.S.

A table compiled from results (3) and (4) with the previous numerical values would give:

θ	40°	30°	20°	10°	0°
Vel.	0	3·69	4·84	5·41	5·59
Accel.	22·34	16·76	11·17	5·59	0

This motion is approx. equal to the actual motion of the pendulum, being a little quicker.

The periodic time of this S.H.M. can easily be shown to be $2\pi\sqrt{\frac{l}{g}}$, and hence the periodic time of the actual motion of the pendulum may be taken as rather more than this.

In the above example, $2\pi\sqrt{\frac{l}{g}} = 1{\cdot}57$ secs. approx.

3. *A closer Approximation to the Actual Motion.*

In the above comparison it has been seen that the velocity in the second motion is greater than the actual velocity of the pendulum. In particular, whilst the greatest velocity of the S.H.M. is $\alpha\sqrt{gl}$, that of the pendulum is only $\sqrt{2gl(1-\cos\alpha)}$.

Now
$$\sqrt{2(1-\cos\alpha)} = \sqrt{2\left(\frac{\alpha^2}{\lfloor 2} - \frac{\alpha^4}{\lfloor 4} + \ldots\right)}$$

$$= \alpha\sqrt{1 - \frac{\alpha^2}{12}},$$

and this may differ considerably from α.

Hence consider a second S.H.M. in which, whilst the amplitude is still $l\alpha$, the max. vel. is exactly that of the pendulum at C, *i.e.* $\sqrt{2gl(1-\cos\alpha)}$. The velocity at P can easily be shown to be $\frac{\sqrt{l^2\alpha^2 - l^2\theta^2}}{l.\alpha}$ of this max. vel.

Hence
$$v^2 = 2gl(1-\cos\alpha).\frac{\alpha^2-\theta^2}{\alpha^2} \quad \text{....................................(5)}$$

and
$$\text{the accel.} = \frac{2g(1-\cos\alpha)}{\alpha^2}.\theta. \quad \text{...(6)}$$

These should now be compared with the previous results.

For example, $$\frac{2(1-\cos\alpha)}{\alpha^2}=1-\frac{\alpha^2}{12}+\dots,$$

and hence the velocity given by (5) is less than that given in (3).

But the velocity given by (5) is greater than that given in (1) if

$$(1-\cos\alpha)\frac{\alpha^2-\theta^2}{\alpha^2}>\cos\theta-\cos\alpha,$$

i.e. if $$\alpha^2-\theta^2-\alpha^2\cos\alpha+\theta^2\cos\alpha>\alpha^2\cos\theta-\alpha^2\cos\alpha,$$

i.e. if $$\alpha^2(1-\cos\theta)>\theta^2(1-\cos\alpha),$$

i.e. if $$\frac{1-\cos\theta}{\theta^2}>\frac{1-\cos\alpha}{\alpha^2}.$$

i.e. if $$\sin\frac{\theta}{2}\Big/\frac{\theta}{2}>\sin\frac{\alpha}{2}\Big/\frac{\alpha}{2},$$

which is true since $\theta<\alpha$ and both $<\pi/2$.

Thus, this third motion can be taken to be intermediate between the other two. The periodic time is easily seen to be

$$\frac{2\pi}{\sqrt{\dfrac{2g(1-\cos\alpha)}{l\alpha^2}}}\quad\text{or}\quad 2\pi\sqrt{\frac{l}{g}}\cdot\frac{\alpha}{2\sin\dfrac{\alpha}{2}}.$$

This value is therefore a nearer approx. to the actual periodic time of the pendulum, but is still a little too small.

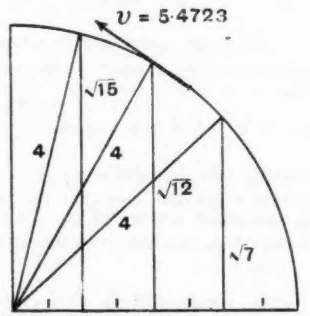


FIG. 2.

In compiling a table of numerical values in this case it is better to proceed from the maximum velocity $\sqrt{2gl(1-\cos\alpha)}$, which can be shown to be $16\sin 20^\circ$ or 5·4723, as follows:

$$\text{Velocity at } X_1=v\times\frac{\sqrt{7}}{4}=3{\cdot}62.$$

$$\text{,, \quad ,, } X_2=v\times\frac{\sqrt{12}}{4}=4{\cdot}74.$$

$$\text{,, \quad ,, } X_3=v\times\frac{\sqrt{15}}{4}=5{\cdot}30.$$

Also max. accel., *i.e.* at A, $=\dfrac{2g(1-\cos\alpha)}{\alpha}=21\cdot447=f.$

$$\text{Accel. at } X_1=\tfrac{3}{4}f=16\cdot09.$$
$$,, \quad ,, \quad X_2=\tfrac{1}{2}f=10\cdot72.$$
$$,, \quad ,, \quad X_3=\tfrac{1}{4}f=\ \ 5\cdot36.$$

Hence table is:

θ	40°	30°	20°	10°	0°
Vel.	0	3·62	4·74	5·30	5·47
Accel.	21·45	16·09	10·72	5·36	0

and the periodic time will be found to be 1·60 secs. approx.

Holloway County Sec. School. F. G. HALL.

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66. Hail, Sacred Nymph ! whose Merits are Divine,
Who like bright stars illustriously do shine.
The Times approach (if right the Muse divine)
When female Honour in its turn shall reign.
Then Aristotle shall grow out of date,
And Euclid's fame share poor Megara's fate.
Sicilia shall her Archimede forget,
And Plato's praise with Athens' honour set,
Ptolemy's name in Egypt shall expire,
While all the world the British Dames admire.
—D. M., *Ladies' Diary*, no. 4.

67. More specially, I appoint that five of the "John Welsh Bursaries" shall be given for best proficiency in Mathematics (I would rather say "in Mathesis," if that were a thing to be judged of from competition), but practically, above all, in pure geometry, such being perennially the symptom, not only of steady application, but of a clear methodic intellect, and offering, in all epochs, good promise for all manner of arts and pursuits. . . . —Carlyle's Bequest of Craigenputtoch to the University of Edinburgh.

68. J'ai pris Kersey pour m'enseigner la Richemetique, à 20ˢ par mois, et il ne vient que sept fois la septmaine ; les arts et les sciences sont bien cheres icy, ils ont besoin d'estre bonnes. The same terms are charged by the master of the violin and the lute.—*Memoirs of the Verney Family*, iii. p. 305. Edmund Verney writes to his father, Sir Ralph, Feb. 23, 1657.

69. If I had the disposition of myself . . . I would within fourteen days goe and live with Mr. Kersie, maby three months or half a yeare for to learn to cast an account, . . . [This did not meet with Sir Ralph's approval, and in May 1659 we find Jack lamenting]: I never learned but very little Arethmeticke, for I never did learne any for wayte nor for measure wᶜʰ ought to bee taught Rule by Rule with the other wᶜʰ is money. I never learned but five Rules ; it is true that I had begunne the 6th which is called yᵉ Rule of 3, but I was never perfect in it ; as for all the other five, although that I have not looked over them these 3 yeares thourouly yett I know that in one day I canne make them all perfect againe. But I hope that whatsoever livelyhood I shall assume to myself (with your consent) I shall bee able to goe through, but it must bee Christo Auxiliante.—*Verney Memoirs.* Jack Verney to Sir Ralph from Kensington, June 1657.

REVIEWS.

Elementary Calculus. By C. H. P. Mayo. Pp. xx + 345 + xxxix. 10s. 1919. (Rivington.)

We cordially recommend Mr. Mayo's introduction to the Calculus to the consideration of teachers who are on the look out for a text-book suitable for pupils who do not propose to specialise in mathematics, and have but a slight mathematical equipment. The student will nowhere feel that he is working at something that is of no practical value, and the carefully selected examples are drawn from a wide field of interest. Graphical illustrations are to the fore throughout. The chapters on maxima and minima, and on Taylor's Theorem, may be specially referred to as evidence of the author's success in developing and illustrating the power of ideas. It is, of course, possible to pick holes here and there, but that will little effect the fact that the boy who has been through this volume will have a very fair general idea of the principles and applications of the subject.

Elementary Mensuration, Constructive Plane Geometry and Numerical Trigonometry. By P. Goyen. Pp. viii + 170. 3s. 6d. 1919. (Macmillan.)

The main idea of the compiler of this little manual is to place in the hands of the child proceeding from the elementary school for a two years' stay in the secondary school such a course of geometrical drawing and "what I may call inductive geometry" as he is likely to master in the time. No one will be found to question the value of accurate drawing, both in its applications and in the part it plays in the unconscious acquisition of geometrical facts. But to spend as much as two years without beginning to reinforce "the test of experience by the test of reason" is a procedure that is open to argument. However that may be, Mr. Goyen has done what he set out to do with the thoroughness we have a right to expect from an "Emeritus Inspector of Schools."

An Arithmetic for Preparatory Schools. By T. Dennis. Pp. xiv + 376. Second Edition. 44s. 6d. 1919. (Bell & Sons.)

This text-book appeals to us from several points of view, and, considering the age of those for whom it is intended, not the least important of these is that it conforms in type to the requirements of the B.A.S.S. Committee on "The Influence of School Books on Eyesight"—a feature in books for both young and old which has been systematically ignored in the past. It is also notable for the careful arrangement and gradation of the exercises, and for the extent of the area over which the compiler has cast his net. So large a proportion of the examples have an intrinsic interest that the intelligence of the child is kept on the alert directly he is acquainted with the simple processes in each successive form of operation.

Modern Geometry. The Straight Line and Circle. By C. V. Durell. Pp. x + 145. 6s. 1920. (Macmillan.)

The experience of ten years has shown the author that his *Course of Plane Geometry for Advanced Students* might be revised and re-issued in a handier form for school purposes. Revision, he tells us, involved changes on such a scale that "it seemed desirable to select even a fresh title." But in spite of the omissions, and the condensations, and the additions, the characteristics of its predecessor are not lost; and above all, the remarkable collection of riders, to which Mr. Durell rightly attaches particular importance, has not suffered, but, as many teachers will feel, has been improved, by the re-casting and supplementing which experience has shown him to be advisable.

Correlated Mathematics for Junior Colleges. By E. Breslich. Pp. xiii + 301. 1.25 $. 1919. (University of Chicago Press.)

This volume applies the ideas developed in the author's text-books of secondary school mathematics, and takes its place after the third of those books. It combines the algebra and analytical geometry taught in the first year at the American Universities. The first three chapters are confined to

the consideration of straight lines and areas. After exercises on determinants, the quadratic function is dealt with by the aid of the parabola, this chapter introducing the ideas of derivative, of maxima and minima, and developing a few of the simple properties of the curve. The next three chapters treat of rational integral functions of degree higher than the second, numerical solutions of equations, the general cubic and bi-quadratic, limits, infinite series, partial fractions, and permutations and combinations. The rest of the book deals with the circle and conic, the general equation of the second degree, and curve tracing. The text is enriched with historical notes, and with reproductions of portraits of eminent mathematicians. The series of which this is the latest volume represents more than the whim of a single writer. It represents the working of a fruitful idea in many minds, the books have individually been constructed in accordance with the experience of many teachers, and the problems it presents of "a course both psychological and administratively practicable have been thrashed out at many 'faculty conferences.'" This in itself constitutes a considerable claim to the attention of the thoughtful teacher.

Number Stories of Long Ago: Number Puzzles before the Log Fire. By DAVID EUGENE SMITH. Pp. vii + 132; iv + 14. 48c. (6d.). 1919. (Messrs. Ginn.)

This charmingly written and charmingly illustrated little book is intended for an elementary school reader, and is probably, what it claims to be, the first story-book written about numbers. That the children will enjoy thus tracing the broad stages in the development of characters and processes is certain, and we are happy to think that few grown-ups, with an hour or even less to spare, will lay it down until they have finished it. At the end of each chapter are easy exercises, and these eventually include a series or number and other puzzles, the solutions of which are given in the second of the two booklets mentioned.

Problems of Cosmogony and Steller Dynamics. By J. H. JEANS. Pp. viii + 293. 21s. net. 1920. (Cambridge University Press.)

The Adams prize for 1917 was awarded to Mr. J. H. Jeans and, supplemented by the results of research in the interim, is published in the handsome volume before us. Darwin, in his "Essay on the Genesis of Double Stars" (1908), was able to state that the study of the forms of equilibrium of rotating liquid is almost complete, "and Jeans has made a good beginning in the investigation of the equilibrium of gaseous stars, but much more remains to be discovered." The subject set for the prize in 1917 was: "The course of evolution of the configurations possible for a rotating and gravitating fluid mass, including the discussion of the stabilities of the various forms."

This, says the author, offered "an excuse not only for putting together my own results in essay form, but also for welding them on to the earlier results obtained in the classical papers of Darwin, Poincaré, and other workers at this problem." The result is a most attractive review of hypotheses from Kant and Laplace to the present day, an account of general dynamical theory, and, of course, the discussion which is the more immediate object of the investigations invited by the Adjudicators of the Prize.

"The main object of the Essay is to build a framework of absolute mathematical truth; the backbone of the structure is the theoretical investigation into the behaviour of rotating masses."

The first chapter deals with the various types of uniformity of structure found in celestial objects, and describes the theory of nebulous origin and the rotational theory which are all that survive of the nebular hypothesis, the tidal action theory identified with the names of Chamberlin and Moulton. Chapter II. deals with general dynamics and criteria of stability, etc., and Chapter III. with the classical configurations of equilibrium of a rotating homogeneous mass—Maclaurin's spheroids and Jacobi's ellipsoids. Chapter IV. provides the material necessary for the quest for such configurations as are of distorted ellipsoidal shape, and Chapter V. brings us to the pear-shaped series and the general conclusion that there are no figures of stable equilibrium except ellipsoids and spheroids. The chapter on cataclysmic motion com-

pletes so far the study of the statical and dynamical motion in the tidal, rotational, and double-star problems. The conclusions are stated as follows: The final result of the dynamical motion appears in every case to be fission into detached masses, although a rigorous mathematical proof of this has not been obtained. In the Tidal Problem any finite number of detached masses may result; in the Rotational Problem the mass appears to divide into two bodies of unequal size; in the Double-Star Problem the mass breaks up into a very great number of small masses. Hence it appears highly probable that tidal action may produce systems such as are seen in our own solar system and in the systems of Jupiter, Saturn, etc.; that increasing rotation may produce systems such as are seen in ordinary binary stars; and that the close approach of two stars revolving about one another may produce systems such as Saturn's rings and possibly the asteroids.

The results for the motion of compressible and non-homogeneous masses are next presented, and here the author finds himself quite at home. The abstract results set forth are now applied to the problems of reality. The evolution of gaseous masses is discussed and leads on to the question of the evolution of rotating nebulae, of star clusters, of binary and multiple stars (with sections on the process of fission and motion subsequent to fission), and the volume closes with a chapter on the origin and evolution of the solar system. Here the section on the tidal theory is of particular interest, for although the theory is one that is "beset with difficulties," and in some respects "definitely unsatisfactory," yet to the author it is "more acceptable than the rotational theory or any other theory so far offered of the genesis of the solar system."

The beautiful photographs of nebulae with which the book is enriched add much to the charm of a volume which is heartily welcome, not only from the masterly researches of the author and his predecessors which it contains, but incidentally from the admirable lesson it affords the reader in the invaluable art of balancing *pros* and *cons*, and in a caution that is almost uncanny in its many-sided wariness.

Mnemonic Notation for Engineering Formulae. Report of the Science Committee of the Concrete Institute. With Explanatory Notes by E. Fiander Etchells. Pp. 116. 6s. net. (Spon & Co.)

"It is submitted that the formulae of science should not be expressed in the misleading symbols which are no more than a tangled chain of accidents. The notation of science should comply with the principles of science and should be the embodiment of organised common sense." The simple principle on which the notation in the Report is built up is that of successive curtailment. Thus "bending moment" becomes B. One letter being insufficient in the case of bending moment at the *end* of a beam, this will be B_E. There is nothing in the proposals more revolutionary than this, and the sole difficulty with which the compiler has had to deal has been to resist the inevitable tendency "to over-do it." It is more particularly for the structural engineer that the booklet is intended.

Catalogues of Rare and Standard Books. Issued by Messrs. Sotheran & Co. 1918-9. (43 Piccadilly, W. 1.)

When a catalogue is something more than a mere catalogue, as has for many years past been the case with those issued by the well-known firm of Sotheran, it is not unseemly that attention should be drawn to the fact. Many of us will remember the old days when we picked up our treasures on stalls or within the murky recesses of Holywell Street, and at least one of us has experienced his delirious moments over a barrow in the Farringdon Road. But Holywell Street and its like, if it had a like, have long passed away, and the reign of the catalogue is supreme. Fifty years ago one could number on two hands at most the firms who sent out with something like regularity to country purchasers enticing summaries of the contents of their loaded shelves. In those days we suppose that it would not have "paid" a firm to send out lists of books confined to the exact and applied sciences. But now, until the Great War, the arrival of mathematical catalogues from France, Germany, and Italy—we have never seen one from across the Atlantic—awakened no

surprise. Although the enjoyment of discovery is all but gone, there is, in these days of high pressure, both pleasure and advantage still left in being able to scan the contents of shelves, however lofty, without moving a step from the comfortable armchair.

"Not like ye wicked Uncle shall we betray disgust by sighings and sobbings,
When he finds himself smother'd with leaves (of fat catalogues) heap'd up by Robins."

Instead of having to trudge home, like a Charles Lamb, folio under arm, we may sit by the fire, write a postcard, *nearly* sure of our prize, and await the vagaries of the post or the railway.

The making of catalogues is nothing new to the Sotherans. Over a hundred years have passed since the representative of an old York family of booksellers of that name started business in London. The great man of the trade in those days was G. Willis, and in 1856 the businesses of Sotheran and Willis were amalgamated, and we believe that the conjunction of these two planets was followed by the issue of a catalogue of 600 pages or more, containing the names of books selected from a stock of half a million. From that time the output has been continuous, and in due course special catalogues of scientific works made their appearance. It is proposed to publish shortly an illustrated library edition of their *Bibliotheca Chemico-Mathematica*, which will be lavishly illustrated, and will run to nearly 1000 pages of text. We hope to have the opportunity of describing this on a future occasion. For the moment we shall confine ourselves to a notice of the most recent of the ordinary catalogues, No. 773, which is certainly rather larger than usual (pp. 256), but this is not surprising, as it takes all science for its province. It includes the libraries of the late Professor Henrici, and of Gilberto Govi (1825-1889) of the University of Naples, and contains over 3000 items, of which about one-third are mathematical. Astronomy, Geodesy, Dialling and Horology account for another 500, and general physics for 700. Special sections are given to general and collected works, to sets of the proceedings of learned societies and to scientific journals.

The catalogues have been produced with meticulous care, and it is the rarest thing to find a misprint—no slight achievement if we consider that the titles of ancient and modern works, whatever their language, are generally given in full. But this is not all. In the majority of cases to each item is appended short commentaries, judiciously selected from unimpeachable authorities, and referring to the scope, or to some peculiarities in connection with the book, to something odd or amusing that it contains, to the part it played in the development of the science, and often to the fact that its existence was unknown to some professed historian of the subject.

Let us open at a random page. Here is Coggeshall's *Art of Practical Measuring by the Sliding Rule . . . with the use of Scamozzi's Lines* (1732), and we are told that the sliding rule herein described was the invention of the author. On Condorcet's *Essais d'Analyse* we have an apt quotation from Mr. W. W. R. Ball, and a suggestion by the compiler that Condorcet was evidently unacquainted with Euler's work on the three bodies. Next comes Coriarius, *Pseudo-Quadratur des Cirkels* (1766), and our informant remarks: "It takes as motto Pop's (*sic*) verses:

'Each might his sev'ral province well command,
Would all but stoop to what they understand,'"

and points out that the work contains a chapter on erroneous proofs (which would almost imply that some of the proofs in the book are not erroneous), and that it was unknown to Poggendorf and to Montucla—he might have added that it may have been unknown to De Morgan, for Coriarius in not mentioned in the *Budget*. On Roger Cotes we have De Morgan and the D.N.B.; on Crelle's Tables we have J. W. L. Glaisher (and we are reminded that to that inveterate hunter of tables Degen's *Canon Pellianus* was *probably* unknown); we have Riccati and Montucla on Cristoforo, and in connection with Dansie's *Mathematical Manual* (1627) we learn that it contains a free translation of Napier's *Rabdologia*, bk. i., unknown to Macdonald, the biographer of the inventor of logarithms, and unnoticed in the British Museum catalogue.

Ex uno disce omnes. We have, we hope, said enough to indicate how extremely useful these catalogues may be to the mathematician, and that is our sole excuse for giving space to a notice of what lies so markedly outside the category of "trade catalogues." We have alluded to the wide reading of the compiler. His own comments are often of the raciest, but we regret to note that on one occasion he allowed himself to be carried away by his patriotic feelings and to make an atrocious remark about a most distinguished living writer. That, however, was not in No. 773, and the momentary lapse must not prevent us from assuring our readers that in these catalogues they will find amusement and instruction for the odd half-hour, and many a useful hint for their more serious undertakings.

CORRESPONDENCE.

23rd November, 1920.

To the Editor of the *Mathematical Gazette*.

Sir,

So much is it a matter of course that questions in mathematics are presented without superfluous data that the advice, "You will know you are wrong if you haven't made use of all you are told," is always accepted as sound. But neither in real life nor in constructive research are the data relevant to each particular problem specified in this convenient way, and I wonder sometimes if there would not be educational value in examples in which the student had to select as well as to use material.

It is true that occasionally the faculty of selection has to be exercised on method, but (i) it is usually to illustrate a given method that an example is set, (ii) if no one method is indicated the student is in fact more concerned to find one tool than to choose between two, (iii) the resemblance between choosing a method and ignoring a hypothesis is somewhat superficial.

Nor is practice in rejection out of place in the training of a specialist in mathematics. Economy is a considerable factor in mathematical elegance, and the elimination of unnecessary assumptions is a recognisable motive in the generalisations of algebra and in the construction of non-euclidean geometries no less than in the most recent developments of formal logic.

Perhaps some of your readers can bring to this subject the light of actual experience or of pedagogical theory. At the risk of an accusation of cowardice, I confess that I am not disposed to advocate either the setting of questions with *insufficient* data or an addition to the horrors of the examination room.—Yours, etc. E. H. NEVILLE.

THE LIBRARY.

CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Please note Librarian's change of address:—C. E. Williams, M.A., 30 Carlton Hill, St. John's Wood, London, N.W. 8.

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